THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1010 I/J University Mathematics 2015-2016
Problem Set 5

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} \cos x & \text{if} \quad x \geq 0; \\ \\ 1 & \text{if} \quad x < 0. \end{cases}$$

   Is $f(x)$ differentiable at $x = 0$?

2. Let $f(x) = x^{2/3}$. Show that $f(x)$ is not differentiable at $x = 0$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} x^3 & \text{if} \quad x \leq 1; \\ \\ ax + b & \text{if} \quad x > 1. \end{cases}$$

   If $f(x)$ differentiable at $x = 1$, find the values of a and b.

4. Let a be a real number and $f(x)$ be a function defined by $f(x) = \lim_{n\to\infty} \dfrac{a(n^x - n^{-x})}{n^x + n^{-x}}$.

   (a) Find $f(0)$.

   (b) Show that $f(x)$ is a constant for $x > 0$ and $f(x)$ is another constant for $x < 0$.

   (c) If $f(x)$ is continuous at $x = 0$, find the value(s) of a.

5. Let $f, g : \mathbb{R} \to \mathbb{R}$ be two functions such that

   - $g(x + y) = g(x)f(y) + f(x)g(y)$ for all $x, y \in \mathbb{R}$
   - $f(0) = 1$, $f'(0) = 0$, $g(0) = 0$ and $g'(0) = 1$

   Show that $g'(x) = f(x)$ for all $x \in \mathbb{R}$.

   (Remark: One may use the above conditions to give a definition of the cosine and sine function by defining them to be $f(x)$ and $g(x)$.)